Metric Spaces and Topology
Lecture 21

Continuous functions to product. Let $Y := \prod_{i\in I} Y_i$ a top space with the product top and let X be a top space. A function $f: X \to Y$ is continuous $\iff \forall i \in I$, $\mathrm{proj}_i \circ f : X \to Y_i$ is continuous. $\boxed{HW}$

Proof. $\Rightarrow$. Trivial because composition of continuous functions is continuous.

$\Leftarrow$. It's enough to show that f-preimages of prebasic open sets of the form $[i \mapsto U_i]$ are open in X. But

$$f^{-1}[i \mapsto U_i] = \{x \in X : f(x)(i) \in U_i\} = \{x \in X : \mathrm{proj}_i \circ f(x) \in U_i\}$$
$$= (\mathrm{proj}_i \circ f)^{-1}(U_i),$$

which is open in X by the continuity of $\mathrm{proj}_i \circ f$. $\square$

Hausdorffness. Product of Hausdorff spaces is Hausdorff.

Proof. Let $X := \prod_{i\in I} X_i$, X_i Hausdorff and let $x, y \in X$ be distinct. $\exists i \in I$ s.t. $x(i) \neq y(i)$, so $\exists$ disjoint open $U_i, V_i \subseteq X_i$ s.t. $U_i \ni x(i)$ and $V_i \ni y(i)$, so $x \in [i \mapsto U_i]$ and $y \in [i \mapsto V_i]$, and $[i \mapsto U_i]$ and $[i \mapsto V_i]$ are disjoint. $\square$

2^{nd} countability. Ctbl product of 2^{nd} ctbl spaces is 2^{nd} ctbl.

Proof. Let $X := \prod_{i \in \mathbb{N}} X_i$ and let $\mathcal{B}_i$ be a ctbl basis for X_i. Let $\mathcal{B}$ consist of all open sets of the form:

$$[i_1 \mapsto U_{i_1}, i_2 \mapsto U_{i_2}, \dots, i_n \mapsto U_{i_n}]$$

where $U_{i_j} \in \mathcal{B}_{i_j}$. Then $\mathcal{B}$ is a basis (check! $\boxed{\text{HW}}$)

$\mathcal{B}$ is ctbl because each set in it is encoded by $(i_1, \dots, i_n) \in \mathbb{N}^{<\mathbb{N}}$ and $(U_{i_1}, \dots, U_{i_n}) \in \mathcal{B}_{i_1} \times \dots \times \mathcal{B}_{i_n}$.

The set of codes is

$$\bigcup_{w \in \mathbb{N}^{<\mathbb{N}}} \mathcal{B}_{w(1)} \times \mathcal{B}_{w(2)} \times \dots \times \mathcal{B}_{w(|w|-1)},$$

which is ctbl being a ctbl union of ctbl sets. $\square$

Metrizability. Ctbl products of (resp. completely) metrizable spaces are (resp. completely) metrizable.

Proof. Let $X := \prod_{i \in \mathbb{N}} X_i$ and let d_i be a *compatible* metric on X_i, i.e. d_i induces the top of X_i. By replacing d_i with $\min(d_i, 1)$, we may assume that $d_i \leq 1$ $\forall i \in \mathbb{N}$.

Let

$$d := \sum_{i \in \mathbb{N}} 2^{-i} d_i.$$

We've shown in HW that this a metric.

It's a $\boxed{\text{HW}}$ exercise to show:
d induces the product topology.

<u>Hint:</u> It is enough to check that for every basic open set U in the product and a point x in it, there is a d-ball centered at x contained in U, and conversely, for any d-ball $B_r(x)$, $\exists$ basic open set $U \ni x$ s.t. $U \subseteq B_r(x)$

We have also shown in HW already that:
If each d_i is complete, then so is d. $\square$

<u>Def.</u> A top space X is called Polish if it is completely metrizable and 2^{nd} ctbl ($\Leftrightarrow$ separable).

<u>Cor.</u> Ctbl product of Polish spaces is Polish.

(Recall also: A subspace of a Polish is Polish $\Leftrightarrow$ it's G_δ.)

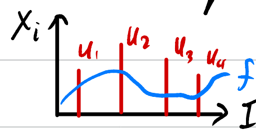
<u>Nonmetrizable spaces.</u> Of course any non-Hausdorff example of top spaces is nonmetrizable (e.g. the half open top on $\{0,1\}$, the cofinite top on $\mathbb{N}$, the Zariski top on $\mathbb{F}^n$ for an ∞ field $\mathbb{F}$),

but perhaps such spaces don't occur naturally in analysis and we would like a Hausdorff example. A batch of such examples are unctbl products of metrizable spaces.

Example. The space of functions $[0,1] \to \mathbb{R}$, i.e. $X := \mathbb{R}^{[0,1]}$, with the pointwise convergence (= product) topology. This is Hausdorff (being a product of Hausdorff spaces) and we show below that it is not metrizable.

Prop. Any unctbl product of nontrivial topological spaces is not 1^{st} ctbl. In particular, it's not metrizable.

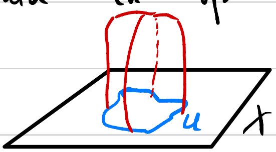
Proof. Let I be an unctbl set & $X := \prod_{i \in I} X_i$ be a product of nontrivial top. spaces X_i. For each $i \in I$, there is an open set $\emptyset \neq U_i \subsetneq X_i$. By Axiom of Choice, $\exists\ f \in \prod_{i \in I} U_i$. We show that f doesn't admit a ctbl basis. Suppose towards a contradiction that $\exists$ a ctbl neighbourhood basis $\mathcal{B}$ at f. Then for each $i \in I$, there is a $B_i \in \mathcal{B}$ (use AC to get $i \mapsto B_i$) with $B_i \subseteq [i \mapsto U_i]$, so by the unctbl-ctbl Pigeonhole Principle, $\exists\ B \in \mathcal{B}$ with an unctbl set $I' \subseteq I$ s.t.



$B \subseteq [i \mapsto U_i)$ for all $i \in I'$. But B contains a basic cylindrical set of the form $[i_1 \mapsto V_1, i_2 \mapsto V_2, \dots, i_n \mapsto V_n]$ $\boxed{\text{HW}}$ and this set is not contained in the intersection $\bigcap_{i \in I'} [i \mapsto U_i]$. □

Existence of continuous real-valued functions.

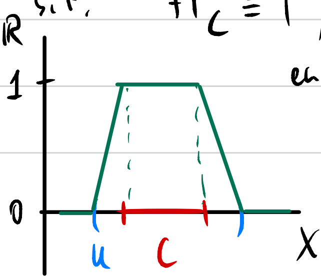
Given a top space X and an open set $U \subseteq X$, we often (say in measure theory) would like $\mathbb{1}_U$ to be continuous.



But this

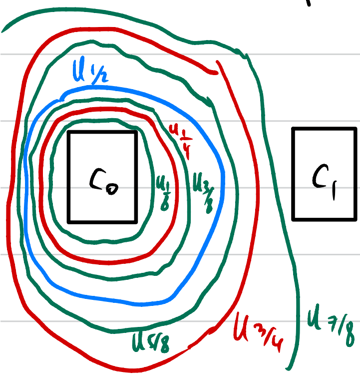
$$\mathbb{1}_U(x) := \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

is true exactly when U is clopen (indeed, the preimages of $(\frac{1}{2}, 1+\varepsilon)$ and $(-\frac{1}{2}, \frac{1}{2})$ should both be open, but the first is U and the second is U^c). There aren't any nontrivial clopen sets in connected spaces like $\mathbb{R}$, this can't happen. The next best thing would be to take a closed subset $C \subseteq U$ ("approximating U"), for example $C := \{x\}$, and try to get a continuous function $f: X \to [0,1]$ s.t. $f|_C \equiv 1$, $f|_{U^c} \equiv 0$. This can be done relatively easily for metric spaces $\boxed{\text{HW}}$, but it turns out that just normality is enough.



Urysohn's Lemma. If X is a normal top space and $C_0, C_1 \subseteq X$ disjoint closed sets, then there is a continuous function $f: X \to [0,1]$ s.t. $f|_{C_0} \equiv 0$ and $f|_{C_1} \equiv 1$.

Proof. We will repeatedly use the following consequence of normality:



(*) $\forall C$ closed, O open, $C \subseteq O \Rightarrow \exists\, U$ open, $C \subseteq U \subseteq_c O$.

Let $D := \left\{ \frac{m}{2^n} : n \in \mathbb{N}^+, m \in \{1, \dots, 2^n - 1\} \right\}$ be the set of dyadic rationals in $(0,1)$ and note that it is dense in $[0,1]$.

Using the normality, we build open sets U_s, $s \in D$, such that $\forall\, s,t \in D$ $s < t \Rightarrow U_s \subseteq_c U_t$, and $\forall s \in D$, $U_s \supseteq C_0$ and $U_s \subseteq_c C_1^c$. Indeed, by (*) $\exists$ open $U_{\frac{1}{2}} \supseteq C_0$ and $\overline{U}_{\frac{1}{2}}$ is still disjoint from C_1 (hence $\exists$ open $V_{\frac{1}{2}} \supseteq C_1$ disjoint from $U_{\frac{1}{2}}$). Next, again by (*), $\exists$ open sets $U_{1/4}$ and $U_{3/4}$ s.t. $C_0 \subseteq U_{\frac{1}{4}} \subseteq_c U_{\frac{1}{2}}$ and $U_{1/2} \subseteq_c U_{3/4} \subseteq_c C_1^c$. And so on...

Let $U_1 := X$. We define $f: X \to [0,1]$ by $x \mapsto \inf$ of $r \in D \cup \{1\}$ s.t. $x \in U_r$. Note that $f|_{C_0} \equiv 0$ and $f|_{C_1} \equiv 1$. It remains to show that f is continuous. Note that since D is dense in $[0,1]$, the sets of the form $[0, a)$ and $(b, 1]$, $a, b \in (0,1)$, form a prebasis, so it's enough to show that the f-preimages of these are open.

Claim 1. $f^{-1}([0,a)) = \bigcup_{r<a, r\in D} U_r$, for any $a \in (0,1)$.

Proof. $\supseteq$. Let $x \in U_r$ with $r<a$. Then by def, $f(x) \leq r < a$, so $x \in f^{-1}([0,a))$.

$\subseteq$. Let $x \in f^{-1}([0,a))$, so $f(x)<a$. By the density of D, $\exists\, r \in D$ s.t. $f(x) < r < a$. Thus, the inf of $s\in D$ s.t. $x \in U_s$ is smaller that r, hence $x \in U_r$. $\boxed{\text{Claim 1}}$

Similarly, one can prove:

Claim 2. $f^{-1}((b,1]) = \bigcup_{r\leq b, r\in D} \overline{U}_r^{\,c}$, for any $b \in (0,1)$.

Thus, f is continuous. $\square$